Computers Math. Applic. Vol. 19, No. 4, pp. 75–88, 1990
Printed in Great Britain. All rights reserved

0097-4943/90 $3.00 + 0.00
Copyright © 1990 Pergamon Press plc

AN ACCURATE SOLUTION OF PARABOLIC EQUATIONS BY EXPANSION IN ULTRASPHERICAL POLYNOMIALS

E. H. Doha

Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

(*Received* 28 *September* 1988; *received for publication* 13 *February* 1989)

Abstract—An ultraspherical expansion technique is applied to numerically obtain the solution of the third boundary value problem for a linear partial differential equation in a one-space variable. The method depends basically on the fact that an expansion in a series of ultraspherical polynomials $C_n^{(\lambda)}$, for the function and its space derivatives occurring in the partial differential equation is assumed, the coefficients of expansion are then determined by reducing the partial differential equation with its boundary and initial conditions to a system of ordinary differential equations for these coefficients. This system may be solved analytically or numerically in a step-by-step manner. The expansion for the function enables one to get the sought-for approximation for any possible value of the parameter $\lambda > -\frac{1}{2}$.

Numerical experiments show that more accurate results may be obtained by taking λ to be small and negative.

The method in its present form may be considered as a generalization of that of Dew and Scraton. The extension of the method to the polar-type equations

$$\frac{\partial u}{\partial t} = p\left[\frac{\partial^2 u}{\partial x^2} + \frac{k}{x}\frac{\partial u}{\partial x}\right]$$

is also considered.

1. INTRODUCTION

Many physical problems require a numerical solution of the heat equation in a one-space variable. The Chebyshev series method has been discussed by many authors, among them Elliott [1], Fox and Parker [2], Knibb and Scraton [3], Dew and Scraton [4] and Doha [5–7].

The essence of the method is that an expansion in Chebyshev series is assumed for the function and its partial derivatives occurring in the partial differential equation, the differential equation with its boundary and initial conditions is reduced to a system of ordinary differential equations in the expansion coefficients.

Our principal aim in the present paper is to discuss the solution of the heat equation

$$\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} \quad (p \text{ is constant}), \tag{1}$$

subject to the boundary conditions

$$\alpha_1 u + \beta_1 \frac{\partial u}{\partial x} = \gamma_1(t), \quad x = 1, \tag{2}$$

$$\alpha_2 u + \beta_2 \frac{\partial u}{\partial x} = \gamma_2(t), \quad x = -1, \tag{3}$$

and the initial condition

$$u = f(x), \quad -1 \leqslant x \leqslant 1; t = 0, \tag{4}$$

but the function $u(x, t)$ and its partial derivatives are expanded in a series of ultraspherical harmonics instead of Chebyshev series.

The ultraspherical polynomials associated with the parameter $\lambda > -\frac{1}{2}$ are a sequence of polynomials $C_n^{(\lambda)}(x)$ $(n = 0, 1, 2, \ldots)$, each, respectively, of degree n, satisfy the orthogonality relation

$$\int_{-1}^{1} (1 - x^2)^{\lambda - \frac{1}{2}} C_m^{(\lambda)}(x) C_n^{(\lambda)}(x)\, \mathrm{d}x = 0. \quad (m \neq n) \tag{5}$$

For our present purposes it is convenient to standardize the ultraspherical polynomials so that

$$C_n^{(\lambda)}(1) = 1 \quad (n = 0, 1, 2, \ldots). \tag{6}$$

This is not the usual standardization, but has the desirable properties that $C_n^{(0)}(x)$ is identical with the Chebyshev polynomial of the first kind $T_n(x)$, $C_n^{(1/2)}(x)$ is the Legendre polynomial $P_n(x)$, and $C_n^{(1)}(x)$ is equal to $(1/(n+1))U_n(x)$, where $U_n(x)$ is the Chebyshev polynomial of the second kind. In this form the polynomials may be generated by using Rodrigue's formula

$$C_n^{(\lambda)}(x) = \frac{(-\frac{1}{2})^n \Gamma(\lambda + \frac{1}{2})}{\Gamma(n + \lambda + \frac{1}{2})} (1 - x^2)^{-\lambda + \frac{1}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (1 - x^2)^{\lambda + n - \frac{1}{2}}. \tag{7}$$

The following two recurrence relations are of fundamental importance in developing the present work. These are

$$(n + 2\lambda) C_{n+1}^{(\lambda)}(x) = 2(n + \lambda) x C_n^{(\lambda)}(x) - n C_{n-1}^{(\lambda)}(x) \tag{8}$$

and

$$2(n + \lambda) C_n^{(\lambda)}(x) = \frac{n + 2\lambda}{n + 1} \frac{\mathrm{d}C_{n+1}^{(\lambda)}(x)}{\mathrm{d}x} - \frac{n}{n + 2\lambda - 1} \frac{\mathrm{d}C_{n-1}^{(\lambda)}(x)}{\mathrm{d}x}. \tag{9}$$

It is to be noted here that the recurrence formula (8) may be used to generate the ultraspherical polynomials starting from $C_0^{(\lambda)}(x) = 1$ and $C_1^{(\lambda)}(x) = x$. The reader is referred to Szegö [8], for a full description of the ultraspherical polynomials.

The motivations for adopting this approach is that the numerical methods for parabolic and hyperbolic partial differential equations, of course, need not be based on Chebyshev polynomials $T_n(x)$ of the first kind $(\lambda = 0)$.

In some applications like the resolution of thin boundary layers, (see for instance Ref. [9]), an expansion in Legendre polynomials $P_n(x)(\lambda = \frac{1}{2})$ may be more appropriate, because such expansion gives an exceedingly good representation of functions that undergo rapid changes in narrow boundary layers. Some other applications for such methods are the numerical studies of turbulent shear flows, and studies of transition in a circular Couette flow and a pipe Poiseuille flow.

It is also worthy to mention that Chebyshev-spectral and Legendre-spectral methods are extremely sensitive to the proper formulation of boundary conditions. When proper boundary conditions are imposed so that the problem is well-posed, the methods yield very accurate results, when improper boundary conditions are applied, the methods are likely to be explosively unstable. An example is given by Gottlieb and Orszag [9], from which they conclude that while the Chebyshev-spectral method is unbounded and algebraically unstable, the Legendre-spectral is semi-bounded and stable.

2. THE METHOD OF SOLUTION

Suppose now we are given a function $y(x)$ which is continuous in the closed interval $[-1, 1]$, then following Clenshaw [10], we write

$$y(x) = \sum_{n=0}^{\infty} a_n C_n^{(\lambda)}(x) \tag{10}$$

and for the kth derivative of $y(x)$,

$$y^{(k)}(x) = \sum_{n=0}^{\infty} a_n^{(k)} C_n^{(\lambda)}(x) \quad (k = 1, 2, \ldots). \tag{11}$$

On differentiating equation (11), and making use of relation (9), it can easily be shown that

$$a_n^{(k)} = \frac{n+2\lambda-1}{2n(n+\lambda-1)} a_{n-1}^{(k+1)} - \frac{n+1}{2(n+\lambda+1)(n+2\lambda)} a_{n+1}^{(k+1)} \quad n \geqslant 1. \tag{12}$$

Throughout this paper we assume that $f(x)$ of equation (4) satisfies the boundary conditions (2) and (3) to make sure that the solution is free of discontinuities. We also assume that $f(x)$ has a series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} f_n C_n^{(\lambda)}(x), \tag{13}$$

which is uniformly convergent in $(-1, 1)$. It then follows that the solution $u(x, t)$ of equation (1) has a series expansion of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) C_n^{(\lambda)}(x). \tag{14}$$

Equations (13) and (14) give $a_n(0) = f_n$, and a numerical solution of equation (1) with its boundary conditions (2) and (3) is obtained by generating tables for $a_n(t)$.

If $a_n'(t)$ denotes the derivative of $a_n(t)$ with respect to t, then we can write

$$\frac{\partial u}{\partial t} = \sum_{n=0}^{\infty} a_n'(t) C_n^{(\lambda)}(x). \tag{15}$$

Now we assume that

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{\infty} a_n^{(2)}(t) C_n^{(\lambda)}(x) \tag{16}$$

and if we satisfy differential equation (1), we get

$$a_n'(t) = p a_n^{(2)}(t), \quad n \geqslant 0. \tag{17}$$

It can easily be deduced from equation (17) and repeated use of equation (12) that

$$a_n(t) = \sum_{i=0}^{\infty} A_{ni} a_i'(t), \quad n \geqslant 2, \tag{18}$$

where

$$\left.\begin{aligned} A_{n,n-2} &= \frac{(n+2\lambda-2)(n+2\lambda-1)}{4pn(n-1)(n+\lambda-2)(n+\lambda-1)}, \\ A_{n,n} &= -\frac{1}{2p(n+\lambda-1)(n+\lambda+1)}, \\ A_{n,n+2} &= \frac{(n+1)(n+2)}{4p(n+\lambda+1)(n+\lambda+2)(n+2\lambda)(n+2\lambda+1)}, \\ A_{n,i} &= 0, \quad \text{unless } i = n, n \pm 2. \end{aligned}\right\} \tag{19}$$

Boundary conditions (2) and (3) can be written as

$$\alpha_1 \sum_{n=0}^{\infty} a_n(t) + \beta_1 \sum_{n=0}^{\infty} \frac{n(n+2\lambda)}{2\lambda+1} a_n(t) = \gamma_1(t),$$

$$\alpha_2 \sum_{n=0}^{\infty} (-1)^n a_n(t) + \beta_2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{n(n+2\lambda)}{2\lambda+1} a_n(t) = \gamma_2(t),$$

which, after some manipulation, can be put into the form

$$a_0(t) + \sum_{n=2}^{\infty} \mu_n a_n(t) = \lambda_1(t), \tag{20}$$

$$a_1(t) + \sum_{n=2}^{\infty} \nu_n a_n(t) = \lambda_2(t), \tag{21}$$

where

$$\mu_n = \left[\left(\alpha_1 + \frac{n(n+2\lambda)}{2\lambda+1}\beta_1\right)(\alpha_2 - \beta_2) + (-1)^n\left(\alpha_2 - \frac{n(n+2\lambda)}{2\lambda+1}\beta_2\right)(\alpha_1 + \beta_1)\right]\Big/\delta$$

$$v_n = \left[\alpha_2\left(\alpha_1 + \frac{n(n+2\lambda)}{2\lambda+1}\beta_1\right) + (-1)^n\alpha_1\left(\alpha_2 - \frac{n(n+2\lambda)}{2\lambda+1}\beta_2\right)\right]\Big/\delta,$$

$$\lambda_1(t) = [(\alpha_2 - \beta_2)\gamma_1(t) + (\alpha_1 + \beta_1)\gamma_2(t)]/\delta,$$

$$\lambda_2(t) = [\alpha_2\gamma_1(t) - \alpha_1\gamma_2(t)]/\delta,$$

where

$$\delta = 2\alpha_1\alpha_2 - \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0.$$

Equations (20) and (21) are true for all t, and may be differentiated with respect to t, the resulting equations can be used to eliminate $a_0'(t)$ and $a_1'(t)$ from equation (18) to give

$$a_n(t) = b_n(t) + \sum_{i=2}^{\infty} B_{ni}a_i'(t), \quad n \geqslant 2, \tag{22}$$

where

$$b_n(t) = A_{n0}\lambda_1'(t) + A_{n1}\lambda_2'(t), \tag{23}$$

$$B_{ni} = A_{ni} - A_{n0}\mu_i - A_{n1}v_i, \tag{24}$$

relations (19) and (24) give

$$\left.\begin{aligned} B_{2i} &= A_{2i} - A_{20}\mu_i, \\ B_{3i} &= A_{3i} - A_{31}v_i, \\ B_{ni} &= A_{ni}, \quad n \geqslant 4. \end{aligned}\right\} \tag{25}$$

It is now necessary to assume that $a_n(t)$ and $a_n'(t)$ are negligible for $n > N$. Equation (22) can then be written in the matrix form as

$$\mathbf{a}(t) = B\mathbf{a}'(t) + \mathbf{b}(t), \tag{26}$$

where

$$\mathbf{a}(t) = \begin{bmatrix} a_2(t) \\ a_3(t) \\ \vdots \\ a_N(t) \end{bmatrix}, \quad B = \begin{bmatrix} B_{22} & B_{23} & \dots & B_{2N} \\ B_{32} & B_{33} & \dots & B_{3N} \\ \vdots & \vdots & & \vdots \\ B_{N2} & B_{N3} & & B_{NN} \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} b_2(t) \\ b_3(t) \\ \vdots \\ b_N(t) \end{bmatrix}.$$

Equation (26), represents a system of nonhomogeneous linear differential equations with constant coefficients. It is then necessary to solve the matrix differential equation (26) subject to the initial condition $a_n(0) = f_n$.

An analytical solution of equation (26) is given explicitly by Bellman and Cooke [11]

$$\mathbf{a}(t) = \exp(B^{-1}t)\left[\mathbf{a}(0) - B^{-1}\int_0^t \exp(B^{-1}s)\mathbf{b}(s)\,\mathrm{d}s\right], \tag{27}$$

where, as usual, $\exp(At) = \Sigma_{k=0}^{\infty}(tA)^k/k$, A being a square matrix.

Another analytical solution to the matrix differential equation (26) can be expressed as Fox and Parker [2]

$$\mathbf{a}(t) = \sum_{i=1}^{N} (\alpha_i \exp(\gamma_i t) + \beta_i)\mathbf{u}_i, \tag{28}$$

where $\mathbf{u}_i$ are the eigenvectors of the matrix B^{-1} with eigenvalues γ_i, assumed distinct, $\mathbf{v}_i$ are the eigenvectors of the transposed matrix $(B^{-1})^T$, and

$$\alpha_i = (\mathbf{v}_i \cdot \mathbf{a}(0))/c_i, \quad \beta_i = (\mathbf{v}_i^T \cdot \boldsymbol{\delta}_i)/c_i,$$

where

$$c_i = (\mathbf{v}_i \cdot \mathbf{u}_i), \quad \boldsymbol{\delta}_i = -B^{-1}\int_0^t \exp(\gamma_i(t-s))\mathbf{b}(s)\,\mathrm{d}s.$$

Formula (27) is impractical if B is a large matrix, while formula (28) is the best when N is small, and the computation is not difficult, and changes in the initial conditions, given by the vector $\mathbf{a}(0)$, are easily incorporated. For large N the determination of the eigenvalues and eigenvectors can make formula (28) an uneconomical method of solution.

If $\mathbf{b}(t)$ is composed of exponential or oscillatory functions, a particular integral of equation (26) can be obtained by elementary means, and the complementary function can be tabulated by means of the step-by-step formula

$$\mathbf{a}_{m+1} = \phi(-\Delta t B^{-1})\mathbf{a}_m, \tag{29}$$

where $t_m = m\,\Delta t$, $\mathbf{a}_m$ is an approximation to $\mathbf{a}(t_m)$, and $\phi(-\Delta t B^{-1})$ is a rational approximation to $\exp(-\Delta t B^{-1})$. The best known rational approximations to $\exp(-z)$ are the Padé one, (see, for instance, Varga [12] and Fairweather [13]).

If $\phi(-\Delta t B^{-1})$ is taken as a Padé approximant with denominator of higher order than the numerator, tabulation using equation (29) is necessarily stable. In particular, the use of the (2, 3) Padé approximant is suggested by Ref. [4], and it is then convenient to evaluate $\phi(-\Delta t B^{-1})$ from the formula

$$\phi(-\Delta t B^{-1}) = I + \Delta t\left\{B - \frac{\Delta t}{2}I + \frac{1}{12}\left[B + \frac{(\Delta t)^2}{60}\left(B - \frac{\Delta t}{10}I\right)^{-1}\right]^{-1}\right\}^{-1}. \tag{30}$$

Some schemes for approximating the solution of equation (26) in which the time variable is descritized have been considered in Doha [5].

3. AN ALTERNATIVE METHOD OF SOLUTION

If we assume that a function $f(x)$ and its second derivative $f''(x)$ have ultraspherical series expansions

$$f(x) = \sum_{n=0}^{\infty} f_n C_n^{(\lambda)}(x) \tag{31}$$

and

$$f''(x) = \sum_{n=0}^{\infty} F_n C_n^{(\lambda)}(x), \tag{32}$$

then, by making use of the recurrence relation (9), we can show that

$$f_n = \frac{(n+2\lambda-2)(n+2\lambda-1)}{4n(n-1)(n+\lambda-2)(n+\lambda-1)}F_{n-2} - \frac{1}{2(n+\lambda-1)(n+\lambda+1)}F_n + \frac{(n+1)(n+2)}{4(n+\lambda+1)(n+\lambda+2)(n+2\lambda)(n+2\lambda+1)}F_{n+2}. \tag{33}$$

It can also be shown that

$$F_n = \frac{(n+\lambda)\Gamma(n+2\lambda)}{n!}\sum_{\substack{i=n+2\\(i-n)\text{ even}}}^{\infty} \frac{i!}{\Gamma(i+2\lambda)}[(i+\lambda)^2-(n+\lambda)^2]f_i. \tag{34}$$

If follows from equation (34) that equation (1) is equivalent to

$$a_n'(t) = \sum_{i=2}^{\infty} C_{ni}a_i(t), \quad n \geqslant 0, \tag{35}$$

where

$$C_{ni} = \frac{p(n+\lambda)\Gamma(n+2\lambda)}{\Gamma(i+2\lambda)} \cdot \frac{i!}{n!}[(i+\lambda)^2 - (n+\lambda)^2], \quad i \geqslant n+2 \text{ and } i-n \text{ even}$$

$$C_{ni} = 0, \qquad \text{otherwise.} \tag{36}$$

Here we propose to assume that $C_{ni}a_i(t)$ can be neglected for $n \geqslant N+3$, and to eliminate the terms $a_{N+1}(t)$, $a_{N+2}(t)$ by making use of the boundary conditions (20) and (21). We adopt this alternative approach because it leads to an equation which is equivalent to equation (26). Now we can write equation (35) as

$$a_n'(t) = \sum_{i=2}^{N+2} C_{ni}a_i(t). \tag{37}$$

This can be written in the split vector form

$$\mathbf{a}'(t) = C\mathbf{a}(t) + H\omega(t), \tag{38}$$

where

$$C = \begin{bmatrix} C_{22} & C_{23} & \dots & C_{2N} \\ C_{32} & C_{33} & \dots & C_{3N} \\ \vdots & \vdots & & \vdots \\ C_{N2} & C_{N3} & \dots & C_{NN} \end{bmatrix}, \quad H = \begin{bmatrix} C_{2,N+1} & C_{2,N+2} \\ C_{3,N+1} & C_{3,N+2} \\ \vdots & \\ C_{N,N+1} & C_{N,N+2} \end{bmatrix}, \quad \omega(t) = \begin{bmatrix} a_{N+1}(t) \\ a_{N+2}(t) \end{bmatrix}.$$

It is worthy to mention here that many of the elements of C, including all those on and below the main diagonal are zero.

If boundary conditions (20) and (21) are differentiated with respect to t, and if $a_n'(t)$ is substituted from equation (37), we have

$$\sum_{n=2}^{N+2} Y_n a_n(t) = \lambda_1'(t), \tag{39}$$

$$\sum_{n=2}^{N+2} Z_n a_n(t) = \lambda_2'(t), \tag{40}$$

where

$$Y_n = C_{0n} + \sum_{i=2}^{n-2} \mu_i C_{in}, \tag{41}$$

$$Z_n = C_{1n} + \sum_{i=2}^{n-2} v_i C_{in}. \tag{42}$$

Equations (39) and (40) may be written in the matrix form as

$$S\mathbf{a}(t) + T\omega(t) = \boldsymbol{\lambda}'(t) \tag{43}$$

where

$$S = \begin{bmatrix} Y_2 & Y_3 & \dots & Y_N \\ Z_2 & Z_3 & \dots & Z_n \end{bmatrix}, \quad T = \begin{bmatrix} Y_{N+1} & Y_{N+2} \\ Z_{N+1} & Z_{N+2} \end{bmatrix}, \quad \boldsymbol{\lambda}'(t) = \begin{bmatrix} \lambda_1'(t) \\ \lambda_2'(t) \end{bmatrix}.$$

Eliminating $\omega(t)$ between equations (38) and (43), we get

$$\mathbf{a}'(t) = (C - HT^{-1}S)\mathbf{a}(t) + HT^{-1}\boldsymbol{\lambda}'(t). \tag{44}$$

Equation (44) is not new. It can be verified that

$$\sum_{i=0}^{N-2} A_{mi}C_{in} = \delta_{mn} \tag{45}$$

and by using equations (25), (41) and (42), it can be deduced that

$$\sum_{i=2}^{N} B_{mi} C_{in} = \delta_{mn} - \delta_{m2} A_{20} Y_n - \delta_{m3} A_{31} Z_n,$$

for $n \leqslant N + 2$. In terms of the matrices defined above, this gives

$$BC = I - AS$$

and

$$BH = -AT,$$

where

$$A = \begin{bmatrix} A_{20} & 0 \\ 0 & A_{31} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

On eliminating A we have

$$B^{-1} = C - HT^{-1}S. \tag{46}$$

Equation (44) is simply equation (26) written differently. The justification for truncating series (35) at $n = N + 2$ in order to obtain equation (37) is simply that we eventually arrive at an equation which is equivalent to equation (26). There is some computational advantage in this alternative approach, in that equation (46) throws some light on the structure of the matrix B which was not previously apparent, and this may be used to simplify the evaluation of $\phi(-\Delta t B^{-1})$.

Following Dew and Scraton [4], and let θ be any scalar parameter, and let the two matrices $c(\theta)$ and $t(\theta)$ be defined by

$$c(\theta) = (I - \theta C)^{-1} \tag{47}$$

and

$$t(\theta) = T + \theta S c(\theta) H. \tag{48}$$

Then it can be shown that

$$(I - \theta B^{-1})^{-1} = c(\theta) - \theta c(\theta) H t(\theta)^{-1} S c(\theta). \tag{49}$$

Therefore the inverse of the matrix $(I - \theta B^{-1})$ can be obtained in terms of the inverse of the triangular matrix $(I - \theta C)$ and the inverse of the 2×2 matrix $t(\theta)$. Thus if $\phi(z)$ is expressed in partial fractions as

$$\phi(z) = \sum_j \frac{L_j}{1 + l_j z}, \tag{50}$$

it follows immediately that

$$\phi(-\Delta t B^{-1}) = \sum_j L_j (I - \Delta t l_j B^{-1})^{-1} \tag{51}$$

in which the matrix inverses on the right-hand side can be found using equation (49), with $\theta = \Delta t l_j$. If ϕ is taken as the (2, 3) Padé approximant, the appropriate values of L_j and l_j ($j = 1, 2, 3$) are given in Ref. [4] as

$$L_1 = 5.0297778578, \quad l_1 = 0.2748888296,$$

$$L_2, L_3 = -2.0148889289 \pm i0.7365075550,$$

$$l_2, l_3 = 0.1625555852 \pm i0.1849493244.$$

Equation (51) may thus be considered as an alternative to formula (30) given above. The real advantage of equation (51) comes when u is known to be an even function of x. In this case every term in C, H, S and T with an odd suffix can be omitted, as well as the entire second row of S

and T, thus S becomes a row matrix, H a column matrix and T and $t(\theta)$ scalars, so that equation (51) is very simple to use.

It is worth to note here that although the method of Section 3 is computationally simpler than that of Section 2, it is mathematically equivalent and will produce identical results. An improvement on the above methods has been described in Ref. [4] by a method which produces more accurate results without any additional computation.

4. EXTENSION TO POLAR-TYPE EQUATIONS

The methods described above can be extended to equations of the form

$$\frac{\partial u}{\partial t} = p\left(\frac{\partial^2 u}{\partial x^2} + \frac{k}{x}\frac{\partial u}{\partial x}\right). \tag{52}$$

For this equation u may be taken as an even function of x, and there is no danger of a singularity at $x = 0$. If we use the same notations of Sections 1 and 2, then satisfaction of equation (52) gives

$$\frac{2n+2\lambda-1}{2n+\lambda-1}[a'_{2n-1} - pa^{(2)}_{2n-1}] + \frac{2n+1}{2n+\lambda-1}[a'_{2n+1} - pa^{(2)}_{2n+1}] = 2pk\, a^{(1)}_{2n}, \tag{53}$$

repeated use of relation (12) and after some manipulation, then equation (53) will take the form

$$\sum_{i=0}^{\infty} A_{m-2,2i} a'_{2i}(t) = \sum_{j=m-2}^{m} T_{m-2,j} a_j(t), \quad m \geqslant 4,\ m \text{ even}, \tag{54}$$

where

$$\left.\begin{aligned}
A_{m-2,m-4} &= \frac{(m+2\lambda-4)(m+2\lambda-2)(m+2\lambda-3)(m+2\lambda-1)}{4pm(m-2)(m-1)(m+\lambda-4)(m+\lambda-3)(m+\lambda-2)},\\
A_{m-2,m-2} &= \frac{(m+2\lambda-2)(m+2\lambda-1)[\lambda(m+2\lambda+1)-(m-1)(m-3)]}{4pm(m-1)(m+\lambda-3)(m+\lambda-2)(m+\lambda-1)(m+\lambda)},\\
A_{m-2,m} &= \frac{(m+\lambda)(m+4\lambda)-(1+3\lambda)}{4p(m+\lambda-2)(m+\lambda-1)(m+\lambda)(m+\lambda+1)},\\
A_{m-2,m+2} &= \frac{(m+1)(m+2)}{4p(m+\lambda)(m+\lambda+1)(m+\lambda+2)(m+2\lambda)},\\
A_{m-2,n} &= 0, \quad \text{otherwise},
\end{aligned}\right\} m \geqslant 4$$

and

$$T_{m-2,n} = \begin{cases} \dfrac{(m+k-2)(m+2\lambda-2)(m+2\lambda-1)}{m(m-1)(m+\lambda-2)}, & n = m-2,\\[2ex] \dfrac{m+2\lambda-k+1}{m+\lambda}, & n = m,\\[2ex] 0, & \text{otherwise}. \end{cases}$$

In the present case equation (20) takes the form

$$a_0(t) + \sum_{i=1}^{\infty} \mu_{2i} a_{2i}(t) = \lambda_1(t). \tag{55}$$

Elimination of $a'_0(t)$ between equations (54) and (55) gives

$$\sum_{j=m-2}^{m} T_{m-2,j}\, a_j(t) = b_{m-2}(t) + \sum_{i=1}^{\infty} B_{m-2,2i} a'_{2i}(t), \tag{56}$$

where

$$b_{m-2}(t) = A_{m-2,0}\lambda_1'(t), \quad B_{m-2,2i} = A_{m-2,2i} - A_{m-2,0}\mu_{2i}.$$

Equation (56) can be written in the finite matrix form

$$T\mathbf{a}(t) = \mathbf{b}(t) + B\mathbf{a}'(t), \tag{57}$$

where

$$\mathbf{a}(t) = [a_2(t), a_4(t), \ldots, a_{2N}(t)]^T, \quad \mathbf{b}(t) = [b_2(t), b_4(t), \ldots, b_{2N}(t)]^T$$

$$T = \begin{bmatrix} T_{2,2} & T_{2,4} & \cdots & T_{2,2N} \\ T_{4,2} & T_{4,4} & \cdots & T_{4,2N} \\ \vdots & \vdots & & \vdots \\ T_{2N,2} & T_{2N,4} & \cdots & T_{2N,2N} \end{bmatrix}, \quad B = \begin{bmatrix} B_{2,2} & B_{2,4} & \cdots & B_{2,2N} \\ B_{4,2} & B_{4,4} & \cdots & B_{4,2N} \\ \vdots & \vdots & & \vdots \\ B_{2N,2} & B_{2N,4} & & B_{2N,2N} \end{bmatrix}$$

The matrix T is a lower triangular nonsingular matrix, then after premultiplying equation (57) by T^{-1} we get again a system of ordinary differential equations like that of equation (26).

The method of Section 3 can also be applied to equation (52). We note that if

$$f(x) = \sum_{n=0}^{\infty} f_n C_n^{(\lambda)}(x),$$

$$\frac{1}{x} f'(x) = \sum_{n=0}^{\infty} g_n C_n^{(\lambda)}(x)$$

and if we define h_n by

$$h_n = \frac{n+2\lambda-2}{n(n+\lambda-2)} g_{n-2} + \frac{1}{n+\lambda} g_n,$$

then

$$f_n = \frac{n+2\lambda-1}{4(n+\lambda-1)} h_n - \frac{(n+1)(n+2)}{4(n+2\lambda)(n+\lambda+1)} h_{n+2}.$$

Now it can be easily shown that

$$h_n = \frac{4(n+\lambda-1)\Gamma(n+2\lambda-1)}{n} \sum_{\substack{j=n \\ m-n \text{ even}}}^{\infty} \frac{j!}{\Gamma(j+2\lambda)} f_j \tag{58}$$

and

$$g_n = (n+\lambda) \sum_{\substack{m=n+2 \\ m-n \text{ even}}}^{\infty} (-1)^{\frac{m-n+2}{2}} \prod_{j=0}^{\frac{m-n+2}{2}} \left(\frac{n+2j+2}{n+2j+2\lambda} \right) h_m.$$

Accordingly, partial differential equation (52) may be reduced to

$$a_n'(t) = \sum_{i=2}^{\infty} C_{ni} a_i(t) + \sum_{i=2}^{\infty} L_{ni} h_i, \quad n \geqslant 0, i, n \text{ even}, \tag{59}$$

where C_{ni} is given by equation (36) and

$$L_{ni} = pk(n+\lambda)(-1)^{(i-n+2)/2} \frac{\frac{i}{2}!\,\Gamma\left(\frac{n}{2}+\lambda\right)}{\frac{n}{2}!\,\Gamma\left(\frac{i}{2}+\lambda\right)}, \quad i \geqslant n+2, i-n \text{ even},$$

$$L_{ni} = 0, \qquad \text{otherwise}.$$

Now, relation (58) may be written in the matrix form

$$h_i = \sum_{j=2}^{\infty} H_{ij} a_j,$$

where

$$H_{ij} = \begin{cases} \dfrac{4(i+\lambda-1)\Gamma(i+2\lambda-1)}{\Gamma(j+2\lambda)} \cdot \dfrac{j!}{i!}, & j \geqslant i, \\ 0, & \text{otherwise.} \end{cases}$$

Hence equation (59) may be written in the form

$$a_n'(t) = \sum_{i=2}^{\infty} K_{ni} a_i(t) \quad n \geqslant 0, i, n \text{ even}, \tag{60}$$

where

$$K_{ni} = C_{ni} + \sum_{\substack{j=2 \\ j \text{ even}}}^{\infty} L_{nj} H_{ji}.$$

Equation (60) is of the form of equation (35), and has the same method of solution.

5. AN IMPROVEMENT OF THE ABOVE METHOD

In the above work $a_n(t)$ is tabulated only for $n = 2, 3, \ldots, N$. Ultimately $a_0(t)$ and $a_1(t)$ can be found using equations (20) and (21). But when the boundary conditions contain the derivative at $x = \pm 1$, μ_n and ν_n are of order n^2, so the neglect of terms such as $\mu_{N+1} a_{N+1}(t)$, $\nu_{N+1} a_{N+1}(t)$ in equations (20) and (21) may lead to loss of accuracy. The above methods are modified so as to take at least the next two terms in these series into account.

Let

$$W = \begin{bmatrix} \mu_{N-1} & \mu_N \\ \nu_{N-1} & \nu_N \end{bmatrix}^{-1} \begin{bmatrix} \mu_{N+1} & \mu_{N+2} \\ \nu_{N+1} & \nu_{N+2} \end{bmatrix} \tag{61}$$

and let $b_{N-1}(t), b_N(t)$ be defined by

$$\begin{bmatrix} b_{N-1}(t) \\ b_N(t) \end{bmatrix} = \begin{bmatrix} a_{N-1}(t) \\ a_N(t) \end{bmatrix} + W\omega(t), \tag{62}$$

where $\omega(t)$ is the same as in Section 3. The effect of this is that equations (20) and (21) may be written as

$$a_0(t) + \sum_{n=2}^{N-2} \mu_n a_n(t) + \mu_{N-1} b_{N-1}(t) + \mu_N b_N(t) = \lambda_1(t), \tag{63}$$

$$a_1(t) + \sum_{n=2}^{N-2} \nu_n a_n(t) + \nu_{N-1} b_{N-1}(t) + \nu_N b_N(t) = \lambda_2(t), \tag{64}$$

with no neglect of terms before $a_{N+3}(t)$. If now $\mathbf{\hat{a}}(t)$ is defined

$$\mathbf{\hat{a}}(t) = \begin{bmatrix} a_2(t) \\ a_3(t) \\ \vdots \\ a_{N-2}(t) \\ b_{N-1}(t) \\ B_N(t) \end{bmatrix}, \tag{65}$$

it can be seen that equations (38) and (43) become

$$\hat{\mathbf{a}}'(t) = C\hat{\mathbf{a}}(t) + \hat{H}\omega(t) \tag{66}$$

and

$$S\hat{\mathbf{a}}(t) + \hat{T}\omega(t) = \boldsymbol{\lambda}'(t), \tag{67}$$

where

$$\hat{H} = H - \begin{bmatrix} C_{2,N-1} & C_{2N} \\ C_{3,N-1} & C_{3N} \\ \vdots & \\ C_{N,N-1} & C_{NN} \end{bmatrix} W. \tag{68}$$

$$\hat{T} = T - \begin{bmatrix} Y_{N-1} & Y_N \\ Z_{N-1} & Z_N \end{bmatrix} W. \tag{69}$$

In deducing equations (66) from (38) it is implicitly assumed that $b'_{N-1}(t) = a'_{N-1}(t)$, $b'_N(t) = a'_N(t)$, which is permissible only if $a'_{N+1}(t)$ and $a'_{N+2}(t)$ are negligible, this means that equations (38) and (66) are not exactly equivalent but only approximately so. Equations (43) and (67) are, however, exactly equivalent. On eliminating $\omega(t)$ between equations (66) and (67) we have

$$\hat{\mathbf{a}}'(t) = (C - HT^{-1}S)\hat{\mathbf{a}}(t) + \hat{H}\hat{T}^{-1}\boldsymbol{\lambda}'(t). \tag{70}$$

This may be regarded as a modified form of equation (44), to which it is approximately equivalent. This gives in general a better representation of the partial differential equation.

If we define

$$\hat{B} = (C - \hat{H}\hat{T}^{-1}S)^{-1}, \tag{71}$$

then equation (70) is equivalent to

$$\hat{\mathbf{a}}(t) = \hat{B}\hat{\mathbf{a}}'(t) - \hat{B}\hat{H}\hat{T}^{-1}\boldsymbol{\lambda}'(t), \tag{72}$$

which may be regarded as a modified form of equation (26). It is easily shown that $\hat{B}$ is identical with B except that the 2×2 matrix

$$\begin{bmatrix} C_{N-1,N+1} & 0 \\ 0 & C_{N,N+2} \end{bmatrix}^{-1} W$$

is added to the four terms on the bottom right-hand corner of B. The determination of $\phi(-\Delta t\hat{B}^{-1})$ can still be carried out by means of equation (51), provided that $\hat{H}$, $\hat{T}$ are substituted for H, T in equations (48) and (49).

Once $\hat{\mathbf{a}}(t)$ has been tabulated, $a_0(t)$ and $a_1(t)$ can be found by means of equations (63) and (64). In order to find $a_{N-1}(t)$, $a_N(t)$ we note that, from equation (67)

$$\omega(t) = T^{-1}(\boldsymbol{\lambda}'(t) - S\mathbf{a}(t)) \tag{73}$$

and from equation (62)

$$\begin{bmatrix} a_{N-1}(t) \\ a_N(t) \end{bmatrix} = \begin{bmatrix} b_{N-1}(t) \\ b_N(t) \end{bmatrix} - W\omega(t). \tag{74}$$

The work of this section simplifies considerably when u is known to be an even function of x. In particular all the 2×2 matrices become scalars.

6. NUMERICAL EXAMPLE AND DISCUSSION

Consider the problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions

$$u\pm\frac{\partial u}{\partial x}=0,\quad x=\pm 1, \tag{75}$$

and the initial condition

$$u=3-x^2,\quad t=0.$$

This problem has been considered by Elliott [1], Dew and Scraton [4], and Doha [6]. The solution is clearly even, and the boundary conditions give

$$\mu_n=\begin{cases}1+\dfrac{n(n+2\lambda)}{1+2\lambda}\\ 0\end{cases};\qquad \nu_n=\begin{cases}0, & n \text{ even},\\ 1+\dfrac{n(n+2\lambda)}{1+2\lambda}, & n \text{ odd},\end{cases}$$

and accordingly equation (20) takes the form

$$a_0(t)+\mu_2a_2(t)+\mu_4a_4(t)+\mu_6a_6(t)+\cdots=0.$$

Taking $N=6$ and omitting odd rows and columns we have $W=(65+18\lambda)/(37+14\lambda)$.

For the sake of comparison, we consider the case $\lambda=0$, which means that the basis of the expansion are Chebyshev polynomials of the first kind. In this case, we have $W=65/37$ and C, H, $\hat{H}$, S, T and $\hat{T}$ as given below.

$$C=\begin{bmatrix}0&48&192\\0&0&120\\0&0&0\end{bmatrix},\quad H=\begin{bmatrix}480\\384\\224\end{bmatrix},\quad \hat{H}=\frac{8}{37}\begin{bmatrix}660\\801\\1036\end{bmatrix},$$

$$S=[4\quad 272\quad 3108],\quad T=[17472],\quad \hat{T}=[12012]$$

and accordingly

$$B=\begin{bmatrix}-\frac{17}{12}&-\frac{101}{24}&-\frac{37}{4}\\[4pt] \frac{1}{48}&-\frac{1}{30}&\frac{1}{80}\\[4pt] 0&\frac{1}{120}&-\frac{1}{70}\end{bmatrix},\quad \hat{B}=\begin{bmatrix}-\frac{17}{12}&-\frac{101}{24}&-\frac{37}{4}\\[4pt] \frac{1}{48}&-\frac{1}{30}&\frac{1}{80}\\[4pt] 0&\frac{1}{120}&-\frac{267}{41440}\end{bmatrix}.$$

If we take ϕ as the (2, 3) Padé approximant and $\Delta t=0.025$, we have

$$\phi(-\Delta t B^{-1})=\begin{bmatrix}0.99536138&0.88239784&2.61536675\\-0.00242316&0.83459383&0.89558406\\-0.00088950&-0.06117639&0.18798860\end{bmatrix}.$$

Table 1. Solution of equation (75) compared with Elliott, Doha and the theoretical solution

t	0.1	0.2	1.0	
$a_0(t)$	2.315090	2.148088	1.187694	P
	2.315090	2.148087	1.187694	E
	2.315083	2.148112	1.187816	D
	2.315090	2.148088	1.187694	T
$a_2(t)$	−0.480833	−0.451173	−0.250842	P
	−0.480836	−0.451175	−0.250842	E
	−0.480844	−0.451178	−0.250867	D
	−0.480832	−0.451174	−0.250842	T
$a_4(t)$	0.004649	0.006217	0.003966	P
	0.004655	0.006216	0.003966	E
	0.004654	0.006217	0.003966	D
	0.004654	0.006216	0.003966	T
$a_6(t)$	0.000298	0.000065	−0.000025	P
	0.000296	0.000068	−0.000025	E
	0.000297	0.000068	−0.000025	D
	0.000295	0.000068	−0.000025	T
$a_8(t)$	−0.000015	−0.000005	0.000000	P
	−0.000015	−0.000006	0.000000	E
	−0.000015	−0.000006	0.000000	D
	−0.000015	−0.000006	0.000000	T

$N = 6, \lambda = 0$.

The values of $a_n(t)$ for $n = 0, 2, 4, 6, 8$ and $t = 0.1, 0.2, 1.0$ are given in Table 1, together with the correct values obtained by Elliott [1] and Doha [6]. It is worth mentioning here that Elliott used an 11×11 matrix, and Doha used 4×4, against the 3×3 matrix used here. Elliott used an interval 0.005 in t, so that five times as many steps were required in his tabulation. The present method represents a very substantial saving in computing time over Elliott's method, and is to be considered as a generalization to that of Dew and Scraton [4].

We give in Table 2, the corresponding values of $a_n(t)$ for $n = 0(2)8$ and $t = 0.1, 0.2, 1.0$ with $\lambda = -0.45, 0.5, 1.0$. Also the numerical results for the case $N = 10$ and $\lambda = -0.45, -0.25, -0.125, 0.0, 0.5, 1.0$ are given in Table 3.

From these tables we see that the results corresponding to the small and negative value of λ are superior to any of the others.

From this we conclude that the expansion based on Chebyshev polynomials of the first kind ($\lambda = 0$) is not always better than ultraspherical series.

To end this paper, we wish to report that the previous method has been applied by the author to the boundary value problem for parabolic equations in two-space variables, the results will be published in a forthcoming paper.

Table 2. Solution of equation (75) for $\lambda = -0.45$, 0.5 and 1

t	0.1	0.2	1.0	λ
	1.9254528908	1.7837510529	0.9854781453	−0.45
$a_0(t)$	2.4750498243	2.2980628831	1.2710449954	0.50
	2.5555078475	2.3736751072	1.3131156340	1.00
	−0.0867934095	−0.0812261719	−0.0451014781	−0.45
$a_2(t)$	−0.6447101065	−0.6063126739	−0.3374726787	0.50
	−0.7282231037	−0.6860843056	−0.38122116557	1.00
	0.0005117244	0.0006645300	0.0004198866	−0.45
$a_4(t)$	0.0082117478	0.0113017218	0.0072768498	0.50
	0.0108879150	0.01153709129	0.0099769337	1.00
	0.0000222475	0.0000045102	−0.0000020520	−0.45
$a_6(t)$	0.0006118807	0.0001304033	−0.0000545286	0.50
	0.0009795470	0.0002134783	−0.0000861941	1.00
	−0.0000011057	−0.0000003582	0.0000000058	−0.45
$a_8(t)$	−0.0000387396	−0.0000133681	0.0000002059	0.50
	−0.0000676802	−0.0000241065	0.0000003642	1.00

$N = 6$.

Table 3. Solution of equation (75) for $\lambda = -0.45, -0.25, -0.125, 0.0, 0.5$ and 1.0

t	0.1	0.2	0.5	1.0	λ
	1.9254516208	1.7837481591	1.4268982178	0.9854781461	−0.450
	2.1559767328	1.9991884327	1.5999588328	1.1050203826	−0.250
$a_0(t)$	2.2468175331	2.0841710209	1.6682572796	1.1521986450	−0.125
	2.3150909547	2.1480877089	1.7196432299	1.1876947897	0.000
	2.4750498224	2.2980628333	1.8403017891	1.2710449953	0.500
	2.5555069673	2.3736747083	1.9012010926	1.3131156472	1.000
	−0.0867935475	−0.0812266188	−0.0652915097	−0.0451014778	−0.450
	−0.3193750084	−0.2992644929	−0.2406978129	−0.1662708228	−0.250
$a_2(t)$	−0.4114203116	−0.3857879055	−0.3103934172	−0.2144183714	−0.125
	−0.4808319916	−0.4511740137	−0.3631157262	−0.2508417152	0.000
	−0.6447100854	−0.6063133652	−0.4885017565	−0.3374726783	0.500
	−0.7282288895	−0.6860844153	−0.5532448271	−0.3822116597	1.000
	0.0005123349	0.0006644122	0.0006051466	0.0004198867	−0.450
	0.0024642803	0.0032392448	0.0029630615	0.0020562450	−0.250
$a_4(t)$	0.0035948765	0.0047637683	0.0043687482	0.0030319930	−0.125
	0.0046538623	0.0062155646	0.0057141606	0.0039660580	0.000
	0.0082189747	0.0112996616	0.0104811478	0.0072768512	0.500
	0.0108957067	0.0153696716	0.0143665592	0.0099769348	1.000
	0.0000240115	0.0000053810	−0.0000027176	−0.0000020629	−0.450
	0.0001336611	0.0000302666	−0.0000149209	−0.0000113417	−0.250
$a_6(t)$	0.0002116047	0.0000482104	−0.0000234311	−0.0000178251	−0.125
	0.0002954582	0.0000677100	−0.0000324634	−0.0000247160	0.000
	0.0006726464	0.0001574364	−0.0000718679	−0.0000548782	0.500
	0.0010866438	0.0002589684	−0.0001133628	−0.0000867916	1.000
	−0.0000010777	−0.0000004415	−0.0000000051	0.0000000058	−0.450
	−0.0000063636	−0.0000026318	−0.0000000305	0.0000000345	−0.250
$a_8(t)$	−0.0000104188	−0.0000043340	−0.0000000507	0.0000000566	−0.125
	−0.0000150118	−0.0000062804	−0.0000000740	0.0000000816	0.000
	−0.0000380509	−0.0000162735	−0.0000001970	0.0000002081	0.500
	−0.0000669109	−0.0000292177	−0.0000003619	0.0000003681	1.000
	−0.0000000002	0.0000000130	0.0000000004	0.0000000000	−0.450
	−0.0000000058	0.0000000812	0.0000000024	−0.0000000001	−0.250
$a_{10}(t)$	−0.0000000144	0.0000001376	0.0000000041	−0.0000000001	−0.125
	−0.0000000281	0.0000002051	0.0000000062	−0.0000000002	1.000
	−0.0000001525	0.0000005869	0.0000000177	−0.0000000004	0.500
	−0.0000004263	0.0000011463	0.0000000347	−0.0000000008	1.000
	0.0000000009	−0.0000000001	0.0000000000	0.0000000000	−0.450
	0.0000000062	−0.0000000006	0.0000000000	0.0000000000	−0.250
$a_{12}(t)$	0.0000000109	−0.0000000011	0.0000000000	0.0000000000	−0.125
	0.0000000169	−0.0000000016	−0.0000000001	0.0000000000	0.000
	0.0000000552	−0.0000000050	−0.0000000002	0.0000000000	0.500
	0.0000001207	−0.0000000105	−0.0000000004	0.0000000000	1.000

$N = 10$.

Acknowledgements—The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

1. D. Elliott, *Proc. Camb. phil. Soc.* **57**, 823 (1961).
2. L. Fox and I. B. Parker, *Chebyshev Polynomials in Numerical Analysis*, p. 169. OUP, London (1972).
3. D. Knibb and R. E. Scraton, *Comput. J.* **14**, 428 (1971).
4. P. M. Dew and R. E. Scraton, *J. Inst. Math. Applic.* **9**, 229 (1972).
5. E. H. Doha, *Ann. Univ. Sci. Bupest Sect. Comput.* **2**, 115 (1979).
6. E. H. Doha, *Arab J. Math.* **4**, 29 (1983).
7. E. H. Doha, *Indian J. Pure appl. Math.* **15**, 865 (1984).
8. G. Szegö, *Orthogonal Polynomials* Vol. 23, p. 58. Am. Math. Soc. Colloquium Publ. (1985).
9. D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*. SIAM, Philadelphia, Ill. (1977).
10. C. W. Clenshaw, *Proc. Camb. phil. Soc.* **53**, 134 (1957).
11. R. Bellmann and K. I. Cooke, *Differential Difference Equations*. Academic Press, New York (1963).
12. R. S. Varga, *Matrix Iterative Analysis*, p. 266. Prentice-Hall, Englewood Cliffs, N.J. (1962).
13. G. Fairweather, *Finite Element Galerkin Methods for Differential Equations*, p.182. Dekker, New York (1978).